Exercise 35

Solve the Blasius problem of an unsteady boundary layer flow in a semi-infinite body of viscous fluid enclosed by an infinite horizontal disk at $z = 0$. The governing equation and the boundary and initial conditions are

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2}, \quad z > 0,\ t > 0,$$
$$u(z,t) = Ut \quad \text{on } z = 0,\ t > 0,$$
$$u(z,t) \to 0 \quad \text{as } z \to \infty,\ t > 0,$$
$$u(z,t) = 0 \quad \text{at } t \le 0,\ z > 0.$$

Explain the significance of the solution.

Solution

The PDE is defined for $t > 0$ and we have an initial condition, so the Laplace transform can be used to solve it. It is defined as

$$\mathcal{L}\{u(z,t)\} = \bar{u}(z,s) = \int_0^t e^{-st} u(z,t)\,dt,$$

which means the derivatives of u with respect to z and t transform as follows.

$$\mathcal{L}\left\{\frac{\partial^n u}{\partial z^n}\right\} = \frac{d^n \bar{u}}{dz^n}$$
$$\mathcal{L}\left\{\frac{\partial u}{\partial t}\right\} = s\bar{u}(z,s) - u(z,0)$$

Take the Laplace transform of both sides of the PDE.

$$\mathcal{L}\{u_t\} = \mathcal{L}\{\nu u_{zz}\}$$

The Laplace transform is a linear operator.

$$\mathcal{L}\{u_t\} = \nu\mathcal{L}\{u_{zz}\}$$

Transform the derivatives with the relations above.

$$s\bar{u}(z,s) - u(z,0) = \nu\frac{d^2\bar{u}}{dz^2}$$

From the initial condition, $u(z,t) = 0$ for $t \le 0$, we have $u(z,0) = 0$.

$$\frac{d^2\bar{u}}{dz^2} = \frac{s}{\nu}\bar{u}(z,s)$$

The PDE has thus been reduced to an ODE whose solution can be written in terms of exponential functions.

$$\bar{u}(z,s) = A(s)e^{\sqrt{\frac{s}{\nu}}z} + B(s)e^{-\sqrt{\frac{s}{\nu}}z}$$

In order to satisfy the condition that $u(z,t) \to 0$ as $z \to \infty$, we require that $A(s) = 0$.

$$\bar{u}(z,s) = B(s)e^{-\sqrt{\frac{s}{\nu}}z}$$

To determine $B(s)$ we have to use the boundary condition at $z = 0$, $u(0,t) = Ut$. Take the Laplace transform of both sides of it.

$$\mathcal{L}\{u(0,t)\} = \mathcal{L}\{Ut\}$$
$$\bar{u}(0,s) = \frac{U}{s^2} \tag{1}$$

Setting $z = 0$ in the formula for $\bar{u}$ and using equation (1), we have

$$\bar{u}(0,s) = B(s) = \frac{U}{s^2}.$$

Thus,

$$\bar{u}(z,s) = \frac{U}{s^2}e^{-\sqrt{\frac{s}{\nu}}z}.$$

Now that we have $\bar{u}(z,s)$, we can get $u(z,t)$ by taking the inverse Laplace transform of it.

$$u(z,t) = \mathcal{L}^{-1}\{\bar{u}(z,s)\}$$

The convolution theorem can be used to write an integral solution for $u(z,t)$. It says that

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

The inverse Laplace transform of the individual functions are

$$\mathcal{L}^{-1}\left\{\frac{U}{s^2}\right\} = Ut$$
$$\mathcal{L}^{-1}\left\{e^{-\sqrt{\frac{s}{\nu}}z}\right\} = \frac{z}{2}\sqrt{\frac{1}{\pi\nu t^3}}e^{-\frac{z^2}{4\nu t}},$$

so by the convolution theorem, we have for $u(z,t)$

$$u(z,t) = \int_0^t U(t-\tau)\frac{z}{2}\sqrt{\frac{1}{\pi\nu\tau^3}}e^{-\frac{z^2}{4\nu\tau}}\,d\tau.$$

Move the constants in front of the integral.

$$u(z,t) = \frac{Uz}{2\sqrt{\pi\nu}}\int_0^t \frac{t-\tau}{\tau^{3/2}}e^{-\frac{z^2}{4\nu\tau}}\,d\tau$$

Split up the integral into two.

$$u(z,t) = \frac{Uz}{2\sqrt{\pi\nu}}\left(t\int_0^t \frac{1}{\tau^{3/2}}e^{-\frac{z^2}{4\nu\tau}}\,d\tau - \int_0^t \frac{1}{\tau^{1/2}}e^{-\frac{z^2}{4\nu\tau}}\,d\tau\right)$$

Make the substitution,

$$p = \frac{z}{\sqrt{4\nu\tau}} \quad\to\quad \tau = \frac{z^2}{4\nu p^2}$$
$$dp = -\frac{z}{4\sqrt{\nu\tau^3}}\,d\tau \quad\to\quad -\frac{4\sqrt{\nu}}{z}\,dp = \frac{1}{\tau^{3/2}}\,d\tau \quad\to\quad -\tau\frac{4\sqrt{\nu}}{z}\,dp = -\frac{z}{\sqrt{\nu}p^2}\,dp = \frac{1}{\tau^{1/2}}\,d\tau,$$

in both integrals.

$$u(z,t) = \frac{Uz}{2\sqrt{\pi\nu}}\left[t\int_{\infty}^{\frac{z}{\sqrt{4\nu t}}} e^{-p^2}\left(-\frac{4\sqrt{\nu}}{z}\,dp\right) - \int_{\infty}^{\frac{z}{\sqrt{4\nu t}}} e^{-p^2}\left(-\frac{z}{\sqrt{\nu}p^2}\,dp\right)\right]$$

Use the minus signs to switch the limits of integration and move the constants in front of the integrals.

$$u(z,t) = \frac{Uz}{2\sqrt{\pi\nu}}\left[\frac{4t\sqrt{\nu}}{z}\int_{\frac{z}{\sqrt{4\nu t}}}^{\infty} e^{-p^2}\,dp - \frac{z}{\sqrt{\nu}}\int_{\frac{z}{\sqrt{4\nu t}}}^{\infty} p^{-2}e^{-p^2}\,dp\right]$$

Distribute the constant in front of the square brackets.

$$u(z,t) = Ut\cdot\frac{2}{\sqrt{\pi}}\int_{\frac{z}{\sqrt{4\nu t}}}^{\infty} e^{-p^2}\,dp - \frac{Uz^2}{2\nu\sqrt{\pi}}\int_{\frac{z}{\sqrt{4\nu t}}}^{\infty} p^{-2}e^{-p^2}\,dp$$

We can write the solution in terms of the complementary error function erfc, a known special function, which is defined as

$$\operatorname{erfc} x = \frac{2}{\sqrt{\pi}}\int_{x}^{\infty} e^{-p^2}\,dp. \tag{2}$$

Rewrite the two integrals as follows.

$$u(z,t) = Ut\operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}}\right) - \frac{Uz^2}{2\nu\sqrt{\pi}}\int_{\frac{z}{\sqrt{4\nu t}}}^{\infty}\frac{d}{dp}\left(-\frac{1}{p}\right)e^{-p^2}\,dp$$

Integrate the second integral by parts.

$$u(z,t) = Ut\operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}}\right) - \frac{Uz^2}{2\nu\sqrt{\pi}}\left[-\left.\frac{e^{-p^2}}{p}\right|_{\frac{z}{\sqrt{4\nu t}}}^{\infty} - \int_{\frac{z}{\sqrt{4\nu t}}}^{\infty}\left(-\frac{1}{p}\right)\frac{d}{dp}e^{-p^2}\,dp\right]$$

Evaluate the derivative in the integrand and plug in the limits into the first term.

$$u(z,t) = Ut\operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}}\right) - \frac{Uz^2}{2\nu\sqrt{\pi}}\left(\frac{\sqrt{4\nu t}}{z}e^{-\frac{z^2}{4\nu t}} - 2\int_{\frac{z}{\sqrt{4\nu t}}}^{\infty} e^{-p^2}\,dp\right)$$

Use equation (2) to write the integral in terms of erfc.

$$u(z,t) = Ut\operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}}\right) - \frac{Uz^2}{2\nu\sqrt{\pi}}\left[\frac{\sqrt{4\nu t}}{z}e^{-\frac{z^2}{4\nu t}} - \sqrt{\pi}\operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}}\right)\right]$$

Simplifying this, we obtain

$$u(z,t) = Ut\left[\left(1+\frac{z^2}{2\nu t}\right)\operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}}\right) - \frac{z}{\sqrt{\nu\pi t}}e^{-\frac{z^2}{4\nu t}}\right].$$

In order to satisfy the last condition, $u(z,t) = 0$ for $t \leq 0$, write the solution as a piecewise function.

$$u(z,t) = \begin{cases} 0 & t \leq 0 \\ Ut\left[\left(1 + \frac{z^2}{2\nu t}\right)\operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}}\right) - \frac{z}{\sqrt{\nu\pi t}}e^{-\frac{z^2}{4\nu t}}\right] & t > 0 \end{cases}$$

$u(z,t)$ can be written compactly with the Heaviside function $H(t)$.

$$u(z,t) = Ut\left[\left(1 + \frac{z^2}{2\nu t}\right)\operatorname{erfc}\left(\frac{z}{\sqrt{4\nu t}}\right) - \frac{z}{\sqrt{\nu\pi t}}e^{-\frac{z^2}{4\nu t}}\right]H(t)$$

The solution at the back of the book does not include $H(t)$ and hence is only valid for $t > 0$.